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A SYSTEMATICS OF DEONTIC ACTION LOGICS BASED ON BOOLEAN ALGEBRA

Abstract. Within the scope of interest of deontic logic, systems in which names of actions are arguments of deontic operators (deontic action logic) have attracted less interest than purely propositional systems. However, in our opinion, they are even more interesting from both theoretical and practical point of view. The fundament for contemporary research was established by K. Segerberg, who introduced his systems of *basic deontic logic of urn model actions* in early 1980s. Nowadays such logics are considered mainly within propositional dynamic logic (PDL). Two approaches can be distinguished: in one of them deontic operators are introduced using dynamic operators and the notion of violation, in the other at least some of them are taken as primitive. The second approach may be further divided into the systems based on Boolean algebra of actions and the systems built on the top of standard PDL.

In the present paper we are interested in the systems of deontic action logic based on Boolean algebra. We present axiomatizations of six systems and set theoretical models for them. We also show the relations among them and the position of some existing theories on the resulting picture. Such a presentation allows the reader to see the spectrum of possibilities of formalization of the subject.

Keywords: Deontic action logic, algebra of actions, Segerberg, Castro and Maibaum.

Introduction

A Deontic Logic of Action [Segerberg, 1982], an article published by K. Segerberg in 1982 was a milestone in the development of deontic logic since the

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1950s when von Wright [von Wright, 1951] and Kalinowski [Kalinowski, 1953] had published their innovative deontic systems. In his article Segerberg proposed two systems: *basic open deontic logic of urn model action* ($\mathcal{B}.\mathcal{O}.\mathcal{D}.$) and *basic closed deontic logic of urn model action* ($\mathcal{B}.\mathcal{C}.\mathcal{D}.$). He also provided two models for both systems and proved the adequacy theorems. Segerberg's work has become a source of inspiration for the deontic first-order theories of Lokhorst [Lockhorst, 1996] and Trypuz [Trypuz, 2008], and deontic logics of action built in connection with Propositional Dynamic Logic (PDL). In the last class of systems, which are perhaps the most developed and explored deontic logics of action nowadays, we can distinguish (i) those in which deontic operators are defined in the Andersonian-Kangerian style by PDL operators and constant V (violation) [Meyer, 1988, Dignum et al., 1996, Hughes and Royakkers, 2008] and (ii) those which are built on PDL and in which (at least some) deontic operators are taken as primitive. The second approach may be further divided into the systems built on top of standard PDL [McCarty, 1983, Meyden, 1996] and the systems based on Boolean algebra of actions [Castro and Maibaum, 2009]. The latter class of systems is especially promising and is the subject of this article. A characteristic feature of the systems in question is a clear distinction between three layers: the deontic layer, the PDL layer and the action layer shaped by Boolean algebra. That distinction is convinient because each of the layers can be studied separetly and freely combined, allowing for a greater flexibility in the design of systems and the ease of choosing the right one for specific applications.

In this paper we analyze the relations between the systems of deontic action logic which are closely related to Segerberg's systems, and the deontic layer of Castro and Maibaum's DPL logic [Castro and Maibaum, 2009]. Some of the systems introduced in this paper are deductively equivalent to existing ones. Some other are such under certain assumptions. Others will be shown to be stronger or weaker from the systems of Segerberg and Castro and Maibaum.

The paper assumes the following structure. In section 1 a basic deontic action logic without obligation will be described. Its extensions are the subject of section 2. In that section the relations of the introduced systems to the ones of Segerberg and Castro and Maibaum will be provided as well. We also demonstrate that the deontic action logic $\mathcal{B}.\mathcal{C}.\mathcal{D}.$ is not included in DPL logic of Castro and Maibaum and vice versa. Moreover it will be shown that both aforementioned systems are stronger than $\mathcal{B}.\mathcal{O}.\mathcal{D}.$ logic of Segerberg and both are contained in (constructed by us in this paper) system $\mathcal{DAL}^3$.

1. Basic Deontic Action Logic without obligation

We introduce a class of deontic action logics without obligation. All of them share the same language. Every logic of the class will be referred by $\mathcal{DAL}^n$, where n is the unique index of the logic.

1.1. Language of $\mathcal{DAL}^n$

The language of $\mathcal{DAL}^n$ can be defined in Backus-Naur notation in the following way:

$$\varphi \ ::= \ \top \mid \alpha = \alpha \mid \mathtt{P}(\alpha) \mid \mathtt{F}(\alpha) \mid \neg\varphi \mid \varphi \wedge \varphi \tag{1}$$

$$\alpha \ ::= \ a_i \mid \mathbf{0} \mid \mathbf{1} \mid \overline{\alpha} \mid \alpha \sqcup \alpha \mid \alpha \sqcap \alpha \tag{2}$$

where a_i belongs to the finite set of *action generators* Act_0, "$\mathbf{0}$" is the impossible action and "$\mathbf{1}$" is the universal action; "$\top$" is an arbitrary tautology, "$\alpha = \beta$" means that α is identical with β; "$\mathtt{P}(\alpha)$"—α is (strongly) permitted; "$\mathtt{F}(\alpha)$"—α is forbidden, "$\alpha \sqcup \beta$"—α or β (a choice between α and β); "$\alpha \sqcap \beta$"—α and β (parallel doing of α and β); "$\overline{\alpha}$"—not α (complement of α). Further, by *Act* we shall understand a set of formulas defined by (2). Obviously $Act_0 \subseteq Act$.

1.2. Algebra of actions

Every deontic action logic $\mathcal{DAL}^n$ contains as its integral part a Boolean algebra of actions from *Act*. Boolean algebra is one of the possible structures that can be used for representing a space of possible actions. It is a simple structure with a significant expressive power[1]. Below we provide a standard list of axioms for that algebra:

$$\alpha \sqcup \beta = \beta \sqcup \alpha, \ \alpha \sqcap \beta = \beta \sqcap \alpha \tag{3}$$

$$(\alpha \sqcup \beta) \sqcup \gamma = \alpha \sqcup (\beta \sqcup \gamma), \ (\alpha \sqcap \beta) \sqcap \gamma = \alpha \sqcap (\beta \sqcap \gamma) \tag{4}$$

$$(\alpha \sqcap \beta) \sqcup \beta = \beta, \ \alpha \sqcap (\alpha \sqcup \beta) = \alpha \tag{5}$$

$$\alpha \sqcap (\beta \sqcup \gamma) = (\alpha \sqcap \beta) \sqcup (\alpha \sqcap \gamma) \tag{6}$$

$$\alpha \sqcup \overline{\alpha} = \mathbf{1}, \ \alpha \sqcap \overline{\alpha} = \mathbf{0} \tag{7}$$

Formulas (3)—(5) are adequate for lattice. (6) is a distributivity condition. Axioms (3)—(7) characterize Boolean algebra.

[1] Adequacy of Boolean algebra as an ontology of action is not discussed in this paper.

It is also reasonable to require that Boolean algebra of actions is not degenerated, formally:

$$\mathbf{0} \neq \mathbf{1} \tag{8}$$

however it is not a necessary assumption for most of $\mathcal{DAL}^n$ systems presented in this paper.

In Boolean algebra we also define order in a standard way:

$$\alpha \sqsubseteq \beta =_{df} \alpha \sqcap \beta = \alpha \tag{9}$$

By "$\alpha \sqsubseteq \beta$" we mean that "α is a component of β" or that "α is included in β". Obviously "$\sqsubseteq$" is reflexive, antisymmetric and transitive. The fact that the set of generators Act_0 is finite implies that the Boolean algebra of actions is atomic.

We shall think about the action generators as *the bricks from which all actions are made up*. Such simple elements can be combined using algebra operators. They can be performed simultaniously ($\sqcap$) or alternativelly, meaning that one of the two actions takes place ($\sqcup$). A complement of action α is such an agent behaviour, that agent does not perform α, doing something else instead. It is worth noting that action generators are the most basic elements in the Boolean action theory only from the perspective of its construction. As we shall see below in the finite Boolean algebra they are other elements which we might be also considered as "basic", although in different sense.

If the set of action generators is finite, then there are some elements of algebra called *atomic actions* such that there is no action between an atom and the impossible action $\mathbf{0}$. It is a combination of all action generators and has a form:

$$\delta_1 \sqcap \cdots \sqcap \delta_n \tag{10}$$

where δ_k is a generator $a_k \in Act_0$ or its complement. It is worth stressing that not all formulas of that form are atoms since some of them may equal $\mathbf{0}$ (cf. formula 49). In formula (10) generators without complements are "parts" or "components" of atomic action and generators with complement are not. If all generators are present with complements we are dealing with a behavior which can be understood as doing nothing.

It is a well known property of atomic Boolean algebra that any element is a sum of atoms. For actions it can be understood intuitively that an arbitrary agent's action is a sum of action atoms (of an agent) at a certain time. Thus action atoms are parts of all action (except **0**). The element **1** is an alternative of all possible behaviors.

Some agent's actions cannot occur together. In that case they are not possible and their parallel doing ($\sqcap$) equals **0**.

1.3. Basic Deontic Action Logic $\mathcal{DAL}^0$

Axiomatization of $\mathcal{DAL}^0$. $\mathcal{DAL}^0$ is axiomatized by the following set of axioms and rules:

- Axioms of Propositional Calculus (in short PC)
- Axioms (3)—(7)
- Identity axioms:

(11) $$\alpha = \alpha$$

(12) $$\alpha = \beta \to (\varphi \to \varphi(\alpha/\beta)),$$

where $\varphi(\alpha/\beta)$ is any sentence obtained from φ by replacing some or all occurrences of α with β

- Specific axioms for deontic operators[2]:

(16) $$\mathtt{P}(\alpha \sqcup \beta) \equiv \mathtt{P}(\alpha) \wedge \mathtt{P}(\beta)$$

(17) $$\mathtt{F}(\alpha \sqcup \beta) \equiv \mathtt{F}(\alpha) \wedge \mathtt{F}(\beta)$$

(18) $$\alpha = \mathbf{0} \equiv \mathtt{F}(\alpha) \wedge \mathtt{P}(\alpha)$$

Axiom (16) says that a choice between two actions is permitted if and only if each of them is permitted. The same applies (17) for forbidden actions. (18) expresses the fact that only the impossible action is at the same time permitted and forbidden.

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[2]Axiom (18) is equivalent with the conjunction of three following formulas (theses):

(13) $$\mathtt{P}(\mathbf{0})$$

(14) $$\mathtt{F}(\mathbf{0})$$

(15) $$\mathtt{P}(\alpha) \wedge \mathtt{F}(\alpha) \to \alpha = \mathbf{0}$$

Intutively, for an action to be permitted (forbidden) means "permitted (forbidden) in any circumstances", i.e. "in combination with any other action". This sense of permissibility was named "strong" in the literature.

We also add to the logic the standard definitions for missing classical operators of PC and sign "$\neq$":

$$\bot =_{df} \neg\top \tag{19}$$

$$\varphi \vee \psi =_{df} \neg(\neg\varphi \wedge \neg\psi) \tag{20}$$

$$\varphi \to \psi =_{df} \neg(\varphi \wedge \neg\psi) \tag{21}$$

$$\varphi \equiv \psi =_{df} \neg(\varphi \wedge \neg\psi) \wedge \neg(\neg\varphi \wedge \psi) \tag{22}$$

$$\alpha \neq \beta =_{df} \neg(\alpha = \beta) \tag{23}$$

Weak permission. In the scope of our framework we also define a concept of weak permission:

$$\mathtt{P}_{we}(\alpha) =_{df} \neg\mathtt{F}(\alpha) \tag{24}$$

An action is weakly permitted if and only if it is not forbidden. In contrast to a (strongly) permitted action, weakly permited one is permited in *some* situations, in combination with *some* other actions.

The meaning of weak permission will be best captured by the conditions of satisfaction for that operator. Syntactically we can provide the way of understanding the concept by the following theses:

$$\neg\mathtt{P}_{we}(\mathbf{0}) \tag{25}$$

$$\mathtt{P}_{we}(\alpha \sqcup \beta) \equiv \mathtt{P}_{we}(\alpha) \vee \mathtt{P}_{we}(\beta) \tag{26}$$

$$\mathtt{P}(\alpha) \wedge \alpha \neq \mathbf{0} \to \mathtt{P}_{we}(\alpha) \tag{27}$$

Some other theses of $\mathcal{DAL}^0$. There is a list of a few self-explanatory theses of $\mathcal{DAL}^0$ below.

$$\mathtt{P}(\beta) \wedge \alpha \sqsubseteq \beta \to \mathtt{P}(\alpha) \tag{28}$$

$$\mathtt{F}(\beta) \wedge \alpha \sqsubseteq \beta \to \mathtt{F}(\alpha) \tag{29}$$

$$\mathtt{P}(\alpha) \to \mathtt{P}(\alpha \sqcap \beta) \tag{30}$$

$$\mathtt{F}(\alpha) \to \mathtt{F}(\alpha \sqcap \beta) \tag{31}$$

$$\mathtt{P}_{we}(\alpha \sqcap \beta) \to \mathtt{P}_{we}(\alpha) \tag{32}$$

$$\mathtt{P}(\alpha) \wedge \mathtt{F}(\beta) \to \alpha \sqcap \beta = \mathbf{0} \tag{33}$$

1.4. Semantics for $\mathcal{DAL}^0$

Deontic action model for $\mathcal{DAL}^0$ is a structure $\mathcal{M} = \langle \mathcal{DAF}, \mathcal{I} \rangle$, where $\mathcal{DAF} = \langle E, Leg, Ill \rangle$ is a *deontic action frame* in which $E = \{e_1, e_2, \ldots, e_n\}$ is a *nonempty* set of possible outcomes (events), Leg and Ill are subsets of E and should be understood as sets of legal and illegal outcomes, respectively. The basic assumption is that there is no outcome which is legal and illegal:

$$Ill \cap Leg = \emptyset \tag{34}$$

$\mathcal{I} : Act \longrightarrow 2^E$ is an interpretation function for $\mathcal{DAF}$ defined as follows:

$$\mathcal{I}(a_i) \subseteq E, \text{ for } a_i \in Act_0 \tag{35}$$

$$\mathcal{I}(\mathbf{0}) = \emptyset \tag{36}$$

$$\mathcal{I}(\mathbf{1}) = E \tag{37}$$

$$\mathcal{I}(\alpha \sqcup \beta) = \mathcal{I}(\alpha) \cup I(\beta) \tag{38}$$

$$\mathcal{I}(\alpha \sqcap \beta) = \mathcal{I}(\alpha) \cap I(\beta) \tag{39}$$

$$\mathcal{I}(\overline{\alpha}) = E \setminus \mathcal{I}(\alpha) \tag{40}$$

Additionally we accept that the interpretation of every atom is a singleton:

$$\overline{\overline{\mathcal{I}(\delta)}} = 1 \tag{41}$$

where δ is an atom of Act. A basic intuition is such that an atomic action corresponding to (a set with) one event/outcome is indeterministic.

From those definitions it is clear that every action generator is interpreted as a set of (its) possible outcomes, the impossible action has no outcomes, the universal action brings about all possible outcomes, operations "$\sqcup$", "$\sqcap$" between actions and "$\bar{\ }$" on a single action are interpreted as set-theoretical operations on interpretations of actions. A class of models defined as above will be represented by $\mathbf{C}^0$.

Satisfaction conditions for the primitive formulas of $\mathcal{DAL}^0$ in any model $\mathcal{M} \in \mathbf{C}^0$ are defined as follows:

$$\begin{array}{lcl}
\mathcal{M} \models \mathtt{P}(\alpha) & \iff & \mathcal{I}(\alpha) \subseteq Leg \\
\mathcal{M} \models \mathtt{F}(\alpha) & \iff & \mathcal{I}(\alpha) \subseteq Ill \\
\mathcal{M} \models \top & & \\
\mathcal{M} \models \alpha = \beta & \iff & \mathcal{I}(\alpha) = \mathcal{I}(\beta) \\
\mathcal{M} \models \neg\varphi & \iff & \mathcal{M} \not\models \varphi \\
\mathcal{M} \models \varphi \wedge \psi & \iff & \mathcal{M} \models \varphi \text{ and } \mathcal{M} \models \psi
\end{array}$$

Action α is strongly permitted iff all of its possible outcomes are legal. It means in practice that if α is permitted, then it is permitted in combination with any action (cf. thesis 30). The same is true for forbiddance.

It can be proved that the satisfaction condition for weak permission takes the form:

$$\mathcal{M} \models \mathtt{P}_{we}(\alpha) \iff \mathcal{I}(\alpha) \cap Leg \neq \emptyset$$

We say that some action α is weakly permitted if and only if (at least) some of its possible outcomes are legal.

Theorem 1. *$\mathcal{DAL}^0$ is sound and complete with respect to class of models $\mathbf{C}^0$.*

Proof. We prove theorem 1 in standard way by showing that each consistent set of formulas has a model. The canonical model and the truth lemma crucial for this kind of proof are introduced below. A very similar proof one can find in [Castro and Maibaum, 2009]. □

Definition 1. *Let Φ be a maximally consistent set of formulas of the language of $\mathcal{DAL}^0$ and $[\alpha]_=$ be an equivalence class of relation $=$, for $\alpha \in Act$. Then a canonical model for this language has the form:*

- $E^\Phi = \{[\alpha]_= : \alpha \textit{ is an atom of } Act\}$
- $\mathcal{I}^\Phi(\alpha) = \{[\beta]_= \in E^\Phi : \beta \sqsubseteq \alpha \in \Phi\}$
- $Leg^\Phi = \bigcup\{\mathcal{I}^\Phi(\alpha) : \mathtt{P}(\alpha) \in \Phi\}$
- $Ill^\Phi = \bigcup\{\mathcal{I}^\Phi(\alpha) : \mathtt{F}(\alpha) \in \Phi\}$

Lemma 1. *$\mathcal{M}^\Phi = \langle \mathcal{DAF}^\Phi, \mathcal{I}^\Phi \rangle$, where $\mathcal{DAF} = \langle E^\Phi, Leg^\Phi, Ill^\Phi \rangle$, belongs to $\mathbf{C}^0$.*

Proof. Lemma 1 we prove by showing that the canonical model satisfies condition *(34)* and *(35)–(41)*. For *(34)*: let's assume that there exists $[\alpha]_= \in E^\Phi$ s.t. $[\alpha]_= \in Leg^\Phi \cap Ill^\Phi$. Then we have $[\alpha]_= \in Leg^\Phi$ and $[\alpha]_= \in Ill^\Phi$. By definitions of the canonical model and thesis *(28)* we obtain that $\mathtt{P}(\alpha) \in \Phi$ and $\mathtt{F}(\alpha) \in \Phi$, which then implies by *(15)* that $\alpha = \mathbf{0} \in \Phi$. The last formula gives contradiction because α, according to our assumption, should be an atom (or an action identical with atom). Conditions *(35)–(40)* are easily provable also for $\mathcal{I}^\Phi$. Condition *(41)* follows immediately from the definitions of $\mathcal{I}^\Phi$ and E^Φ. □

Lemma 2. $\forall \alpha \in Act$, $\forall [\beta]_= \in \mathcal{I}^\Phi(\alpha)$ $(\beta \sqsubseteq \alpha \in \Phi)$

Proof. The proof of that lemma is inductive, assuming that α can have the forms: $\alpha = a_i$, $\alpha = \mathbf{0}$, $\alpha = \beta \sqcup \gamma$, $\alpha = \beta \sqcap \gamma$, $\alpha = \overline{\beta}$ (cf. the proof of lemma 1 in [Castro and Maibaum, 2009]) $\square$

Lemma 3 (Truth lemma). $\mathcal{M}^\Phi \models \varphi \iff \varphi \in \Phi$

Proof. The proof is inductive. For PL operators the proof is standard. For the other ones we prove as follows:

- $\mathcal{M}^\Phi \models \alpha = \beta \iff \alpha = \beta \in \Phi$
  ($\Longrightarrow$) Assume that $\mathcal{M}^\Phi \models \alpha = \beta$. Then $\mathcal{I}^\Phi(\alpha) = \mathcal{I}^\Phi(\beta)$. For $\mathcal{I}^\Phi(\alpha) = \mathcal{I}^\Phi(\beta) = \emptyset$ we get $\alpha = \mathbf{0} \in \Phi$ and $\beta = \mathbf{0} \in \Phi$ and finally that $\alpha = \beta \in \Phi$. For $\mathcal{I}^\Phi(\alpha)$ and $\mathcal{I}^\Phi(\beta)$ being nonempty sets, we shall notice that they have the same elements, which are all atoms "included" in α and also in β. Let χ be a sum of all atoms γ_k for which it is true that $[\gamma_k]_= \in \mathcal{I}^\Phi(\alpha)$ and $[\gamma_k]_= \in \mathcal{I}^\Phi(\beta)$. Then obviously: $\chi = \alpha \in \Phi$ and $\chi = \beta \in \Phi$ and finally $\alpha = \beta \in \Phi$.
  ($\Longleftarrow$) Assume that $\alpha = \beta \in \Phi$. Then $\alpha \sqsubseteq \beta \in \Phi$ and $\beta \sqsubseteq \alpha \in \Phi$. If so, all atoms "included" in α are "included" in β and vice versa. The last implies that $\mathcal{I}^\Phi(\alpha) = \mathcal{I}^\Phi(\beta)$. Finally $\mathcal{M}^\Phi \models \alpha = \beta$.
- $\mathcal{M}^\Phi \models \mathtt{P}(\alpha) \iff \mathtt{P}(\alpha) \in \Phi$
  $\mathcal{M}^\Phi \models \mathtt{P}(\alpha) \iff \mathcal{I}^\Phi(\alpha) \subseteq Leg^\Phi \iff \mathtt{P}(\alpha) \in \Phi$
- $\mathcal{M}^\Phi \models \mathtt{F}(\alpha) \iff \mathtt{F}(\alpha) \in \Phi$
  $\mathcal{M}^\Phi \models \mathtt{F}(\alpha) \iff \mathcal{I}^\Phi(\alpha) \subseteq Ill^\Phi \iff \mathtt{F}(\alpha) \in \Phi$ $\square$

2. Extensions of $\mathcal{DAL}^0$

Differences among the systems considered in this section lie in two aspects that are intuitively significant: the level of *closeness* of a deontic action logic and the *possibility of performing no action at all.*

In deontic action logic closeness means that for a class of actions, an action has to be either permitted or forbidden. The principle has been present in the philosophy of law at least since Thomas Hobbes, who stated in 17th century that what is not explicitly forbidden by law is permitted. In a more practical context of computer science it is also important that any possible action of an agent is either permitted or forbidden. Some remarks on that subject can be found in ([Segerberg, 1981]).

2.1. $\mathcal{DAL}^1$

Definition 2. *$\mathcal{DAL}^1$ is $\mathcal{DAL}^0$ extended with Generator Closure Axiom:*

$$\mathtt{F}(a_i) \vee \mathtt{P}(a_i), \ \textit{for } a_i \in Act_0 \tag{42}$$

In that system it is accepted that closeness applies to action generators: every action generator is either forbidden or permitted, *no matter their context*. We find such a constraint somehow paradoxical. The paradox can be seen better when we combine (42) with thesis (33). As a result we obtain the principle that it is impossible to perform simultaneously simple elements of agent's behavior freely, but permitted elements can be combined with permitted ones only and forbidden with forbidden ones.

$$a_i \sqcap a_j \neq \mathbf{0} \rightarrow (\mathtt{P}(a_i) \wedge \mathtt{P}(a_j)) \vee (\mathtt{F}(a_i) \wedge \mathtt{F}(a_j)), \ \text{for } a_i, a_j \in Act_0 \tag{43}$$

Theorem 2. *$\mathcal{DAL}^1$ is sound and complete with respect to the class of model $\mathbf{C}^0$ satisfying additionally the following condition for $\mathcal{I}$:*

$$\forall a_i \in Act_0, \ (\mathcal{I}(a_i) \subseteq Leg \textit{ or } \mathcal{I}(a_i) \subseteq Ill) \tag{44}$$

Proof. We need to show that a canonical model for $\mathcal{DAL}^1$ satisfies condition *(44)*. Let us take any generator $a_i \in Act_0$. By *(42)* we know that $\mathtt{F}(a_i) \vee \mathtt{P}(a_i) \in \Phi$, i.e. that $\mathtt{F}(a_i) \in \Phi$ or $\mathtt{P}(a_i) \in \Phi$. Assume that $\mathtt{P}(a_i) \in \Phi$. Then $\mathcal{I}^\Phi(a_i) \in \{\mathcal{I}^\Phi(\alpha) : \mathtt{P}(\alpha) \in \Phi\}$ and obviously $\mathcal{I}^\Phi(a_i) \subseteq Leg^\Phi$. Finally $\mathcal{I}^\Phi(a_i) \subseteq Ill^\Phi$ or $\mathcal{I}(a_i) \subseteq Leg^\Phi$. We obtain the same result assuming that $\mathtt{F}(a_i) \in \Phi$. □

2.2. $\mathcal{DAL}^2$

Definition 3. *$\mathcal{DAL}^2$ is $\mathcal{DAL}^1$ extended with Closure of Total Refraining Axiom:*

$$\mathtt{P}(\overline{a_1} \sqcap ... \sqcap \overline{a_n}) \vee \mathtt{F}(\overline{a_1} \sqcap ... \sqcap \overline{a_n}), \ \textit{where } \{a_1, ..., a_n\} = Act_0 \tag{45}$$

In $\mathcal{DAL}^2$ performing no action at all is either permitted or forbidden. In other words, doing nothing has always some deontic value. In this system it can be proved that every atomic action is either permitted or forbidden:

$$\mathtt{P}(\delta_1 \sqcap \cdots \sqcap \delta_n) \vee \mathtt{F}(\delta_1 \sqcap \cdots \sqcap \delta_n) \tag{46}$$

where δ_k is a generator $a_k \in Act_0$ with or without complement (see proof 2.6 below).

THEOREM 3. *$\mathcal{DAL}^2$ is sound and complete with respect to the class of model* $\mathbf{C}^0$ *satisfying condition* (44) *and the following one:*

$$(E \setminus \bigcup_{a_i \in Act_0} \mathcal{I}(a_i)) \subseteq Leg \text{ or } (E \setminus \bigcup_{a_i \in Act_0} \mathcal{I}(a_i)) \subseteq Ill \tag{47}$$

PROOF. See proof 2.1. □

2.3. $\mathcal{DAL}^3$

DEFINITION 4. *$\mathcal{DAL}^3$ is $\mathcal{DAL}^1$ extended with axiom of Universal Sum of Generators:*

$$(a_1 \sqcup \cdots \sqcup a_n) = \mathbf{1} \tag{48}$$

The question which arises while analysing that system concerns the possibility of performing no action at all. It is worth noting that the axiom (48) is equivalent to:

$$(\overline{a_1} \sqcap \cdots \sqcap \overline{a_n}) = \mathbf{0} \tag{49}$$

It means that it is impossible to refrain from performing any action generator at a certain moment. In other words it is necessary to perform at least one atomic action any time. An agent must do something. Whether that is true or not clearly depends on application. In general it does not seem to be necessary. However, in computer science the action *skip* (which means do nothing) is widely used. If such action is present in a set of atomic actions axiom (48) should be accepted.

The weaker version of the above is admitting that performing no action is either forbidden or permitted—cf. axiom (45).

THEOREM 4. *$\mathcal{DAL}^3$ is sound and complete with respect to the class of model* $\mathbf{C}^0$ *satisfying* (44) *and the following condition:*

$$E = \bigcup_{a_i \in Act_0} \mathcal{I}(a_i) \tag{50}$$

(50) *states that there is not event/outcome which would not belong to any interpretation of action.*

PROOF. We need to show that a canonical model for $\mathcal{DAL}^3$ satisfies condition *(50)*. We know that $(a_1 \sqcup \cdots \sqcup a_n) = \mathbf{1} \in \Phi$. Then $(a_1 \sqcup \cdots \sqcup a_n) \sqsubseteq \mathbf{1} \in \Phi$ and $\mathbf{1} \sqsubseteq (a_1 \sqcup \cdots \sqcup a_n) \in \Phi$. All atoms "included" in $a_1 \sqcup \cdots \sqcup a_n$ are also "included" in $\mathbf{1}$ and vice versa. Thus, taking into account the definition of $\mathcal{I}^\Phi$, it is the case that $\mathcal{I}^\Phi(a_1 \sqcup \cdots \sqcup a_n) = \mathcal{I}^\Phi(\mathbf{1})$, what directly gives *(50)*. □

2.4. $\mathcal{DAL}^4$

Definition 5. *$\mathcal{DAL}^4$ is $\mathcal{DAL}^0$ extended with Atom Closure Axiom:*

$$\mathtt{F}(\delta) \vee \mathtt{P}(\delta), \tag{51}$$

for δ being an atom of algebra.

In $\mathcal{DAL}^4$ closeness applies for atomic actions: every atomic action is either forbidden or permitted. (51) is stronger than (42) since it implies that any possible behavior of an agent in a certain situation is either permitted or forbidden.

Theorem 5. *$\mathcal{DAL}^4$ is sound and complete with respect to the class of model $\mathbf{C}^0$ satisfying the following closure condition:*

$$E = Leg \cup Ill \tag{52}$$

Proof. We need to show that a canonical model for $\mathcal{DAL}^4$ satisfies condition *(52)*. Assume that it is not the case, i.e. that there is some $[\delta]_= \in E^\Phi$ s.t. neither $[\delta]_= \in Leg^\Phi$ nor $[\delta]_= \in Ill^\Phi$. We know that $\mathtt{F}(\delta) \vee \mathtt{P}(\delta) \in \Phi$, i.e. that $\mathtt{F}(\delta) \in \Phi$ or $\mathtt{P}(\delta) \in \Phi$. Then $\mathcal{I}^\Phi(\delta) \subseteq Leg^\Phi$ or $\mathcal{I}^\Phi(\delta) \subseteq Ill^\Phi$. Because $\mathcal{I}^\Phi$ satisfies condition *(41)* (interpretation of every atom is a singleton), $[\delta]_= \in \mathcal{I}^\Phi$. Thus $[\delta]_= \in Leg^\Phi$ or $[\delta]_= \in Ill^\Phi$, which contradicts our assumption. □

2.5. $\mathcal{DAL}^5$

$\mathcal{DAL}^5$ is $\mathcal{DAL}^4$ extended with axiom (48). In that system it is assumed that for every agent it is necessary to perform at least one atomic action at any time and that any possible behavior of an agent is either permitted or forbidden.

Theorem 6. *$\mathcal{DAL}^5$ is sound and complete with respect to the class of model $\mathbf{C}^0$ satisfying conditions* (50) *and* (52)*.*

Proof. See the proofs 2.3 and 2.4. □

2.6. Relations between the systems

It is easy to see that the following relations hold between the systems ("$\subset$" stands for proper inclusion):

- $\mathcal{DAL}^0 \subset \mathcal{DAL}^1$
- $\mathcal{DAL}^0 \subset \mathcal{DAL}^4$
- $\mathcal{DAL}^1 \subset \mathcal{DAL}^2$
- $\mathcal{DAL}^2 \subset \mathcal{DAL}^3$
- $\mathcal{DAL}^4 \subset \mathcal{DAL}^5$

What is less trivial and what we are going to show below is that

- $\mathcal{DAL}^4 \subset \mathcal{DAL}^2$

From that fact it immediately follows that

- $\mathcal{DAL}^5 \subset \mathcal{DAL}^3$

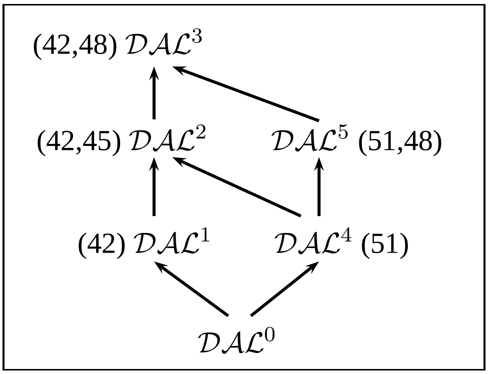

Figure 1. A lattice of $\mathcal{DAL}^n$ systems

For showing that $\mathcal{DAL}^4 \subset \mathcal{DAL}^2$ we need to prove that 51 is a thesis of $\mathcal{DAL}^2$.

Proof. Every atom has a form

$$\delta_1 \sqcap \cdots \sqcap \delta_n$$

where δ_k is a generator $a_k \in Act_0$ or its complement. Of course there is only one atom whose all generators appear as complements: $\overline{a_1} \sqcap ... \sqcap \overline{a_n}$. From *(45)* we immediately get that

$$\mathtt{F}(\overline{a_1} \sqcap ... \sqcap \overline{a_n}) \vee \mathtt{P}(\overline{a_1} \sqcap ... \sqcap \overline{a_n})$$

Other atoms contain at least one generator a_k appearing not as a complement. So, let γ be any atom in which some a_k appears not as a complement. It is obvious that $\gamma \sqsubseteq a_k$ (cf. *(9)*). Then by axiom *(42)*, theses *(28)* and *(29)* and PL we get that $\mathtt{F}(\gamma) \vee \mathtt{P}(\gamma)$, what ends the proof. □

As mentioned earlier, all inclusions of the considered systems are proper. It can be proved by showing that additional axioms of stronger systems are false in some models of weaker ones. As an example we will show, that $\mathcal{DAL}^1$ is not included in $\mathcal{DAL}^5$ and vice versa.

Lemma 4. (42) *is not a thesis of* $\mathcal{DAL}^5$ *and* (51) *is not a thesis of* $\mathcal{DAL}^1$.

Proof. We use the completeness proofs for both logics in question and show (i) that *(42)* is not valid in the class of model adequate for $\mathcal{DAL}^5$ and (ii) that *(51)* is not valid in the class of models adequate for $\mathcal{DAL}^1$.
In the first case we need to show that it is not the case that $\mathcal{I}(a_i) \subseteq Leg$ or $\mathcal{I}(a_i) \subseteq Ill$, for an $a_i \in Act_0$, in a model belonging to the class of models $\mathbf{C}^0$ satisfying conditions *(50)* and *(52)*. For that purpose let us take any model from $\mathbf{C}^0$, satisfying conditions *(50)* and *(52)* such that $\mathcal{I}(a_i)$ is a sum of at least two singletons from which one is a subset of *Leg* and the other is a subset of *Ill*. $\mathcal{I}(a_i)$ is neither a subset of *Leg* nor *Ill*, what was to be proved.
In the second case we need to show that it is not the case that $\mathcal{I}(\delta) \subseteq Leg$ or $\mathcal{I}(\delta) \subseteq Ill$, for atomic action δ, in a model belonging to $\mathbf{C}^0$, satisfying condition *(44)*. For that purpose let us take any model from $\mathbf{C}^0$ satisfying condition *(44)* and, additionally, the property: $E \setminus \bigcup_{a_i \in Act_0} \mathcal{I}(a_i)$ is neither a subset of *Leg* nor *Ill*. Let $\delta = \overline{a_1} \sqcap ... \sqcap \overline{a_n}$ (where $\{a_1, \dots, a_n\} = Act_0$). Then $\mathcal{I}(\delta) = E \setminus \bigcup_{a_i \in Act_0} \mathcal{I}(a_i)$. Thus $\mathcal{I}(\delta)$ is neither a subset of *Leg* nor *Ill*, what was to be proved. □

2.7. $\mathcal{DAL}^n$ and existing systems

C&M. System $\mathcal{DAL}^5$ is equivalent to deontic layer of system DPL of Castro and Maibaum [Castro and Maibaum, 2009] axiomatized by axioms of PC, axioms (3)—(7), identity axioms: (11) and (12), axiom (49) and the following set of axioms:

(53) $$P(\mathbf{0})$$

(54) $$P(\alpha \sqcup \beta) \equiv P(\alpha) \wedge P(\beta)$$

(55) $$P(\alpha) \vee P(\beta) \to P(\alpha \sqcap \beta)$$

$$\neg P_{we}(\mathbf{0}) \tag{56}$$

$$P_{we}(\alpha \sqcup \beta) \equiv P_{we}(\alpha) \vee P_{we}(\beta) \tag{57}$$

$$P_{we}(\alpha \sqcap \beta) \to P_{we}(\alpha) \wedge P_{we}(\beta) \tag{58}$$

$$P(\alpha) \wedge \alpha \neq \mathbf{0} \to P_{we}(\alpha) \tag{59}$$

$$P_{we}(\alpha) \to P(\alpha), \text{where } [\alpha]_{BA} \in at(Act/\Phi_{BA})^3 \tag{60}$$

To show equivalence we assume that operators P and P_{we} correspond to operators $\mathtt{P}$ and $\mathtt{P}_{we}$ respectively. Then we shall notice that "axioms" (55) and (58) of DPL are in fact dependent on the other axioms of the system, i.e. they are provable. Their proofs go as follows[4]:

Proof.

1.	$\mathtt{P}(\alpha) \to \mathtt{P}(\alpha \sqcap \beta)$	Th.30
2.	$\mathtt{P}(\beta) \to \mathtt{P}(\beta \sqcap \alpha)$	Th.30
3.	$\beta \sqcap \alpha = \alpha \sqcap \beta$	Ax.3
4.	$\mathtt{P}(\beta) \to \mathtt{P}(\alpha \sqcap \beta)$	MP: Ax.12, 2, 3
5.	$\mathtt{P}(\alpha) \vee \mathtt{P}(\beta) \to \mathtt{P}(\alpha \sqcap \beta)$	PL:1,4

□

Proof.

1.	$\mathtt{P}_{we}(\alpha \sqcap \beta) \to \mathtt{P}_{we}(\alpha)$	Th.32
2.	$\mathtt{P}_{we}(\beta \sqcap \alpha) \to \mathtt{P}_{we}(\beta)$	Th.32
3.	$\beta \sqcap \alpha = \alpha \sqcap \beta$	Ax.3
4.	$\mathtt{P}_{we}(\alpha \sqcap \beta) \to \mathtt{P}_{we}(\alpha)$	MP: Ax.12, 2, 3
5.	$\mathtt{P}_{we}(\alpha \sqcap \beta) \to \mathtt{P}_{we}(\alpha) \wedge \mathtt{P}_{we}(\beta)$	PL:1,4

□

Finally let us consider axiom (60) of DPL. DPL contains axioms for Boolean Algebra with equality. Thus it includes the extensionality for identity axiom. That together with identity axioms and axiom (51) is equivalent to (60).

Segerberg. Systems $\mathcal{DAL}^0$ and $\mathcal{DAL}^1$ correspond to Segerberg's systems B.O.D. and B.C.D., respectively [Segerberg, 1982], under the assumption that Boolean algebra of B.O.D. and B.C.D. is finite (originally Segerberg assumes it otherwise). Corresponding systems have the same axioms providing that $\mathtt{P}$ and $\mathtt{F}$ operators are Segerberg's operators *Perm* and *Forb*, respectively.

[4]Of course theses 30 and 32 of $\mathcal{DAL}^5$ are also provable for DPL.

As a result of those analyses we obtain that neither system B.C.D ($\mathcal{DAL}^1$) of Segerberg (with finite Boolean algebra) is contained in DPL ($\mathcal{DAL}^5$) of Castro and Maibaum nor vice versa (see lemma 4). Moreover both systems are stronger than B.O.D. ($\mathcal{DAL}^0$) of Segerberg and both are contained in system $\mathcal{DAL}^3$.

Conclusion and future perspectives

We have presented several systems of deontic logic of action and relations between those systems. That presentation allows us to see the spectrum of possibilities of formalizations of the subject. It also makes it possible to understand the relations between the systems already present in the literature.

Future works will be directed towards a providing satisfying account for obligation. Then we are going to investigate the relations between deontic and PDL operators.

Moreover, it is worth noting that Boolean algebra in the considered systems plays the role of an ontology (a formal description) of actions. As it is a very simple ontology it does not reflect many interesting properties of actions (cf. [Trypuz, 2007]). It would be interesting to incorporate more complex description of actions into the area of deontic logic of action.

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